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Assigned: Friday, October 26

Due: Friday, November 2

Note: Please hand in problems on separate sheets of paper

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## Problem 1: Look-back to Lectures S4 and S5 (10 points)

- (a) In Lecture S4 we derived Duhamel's superposition integral for an LTI system:

$$y(t) = h_s(t)u(0) + \int_0^\infty h_s(t-\tau) \frac{du(\tau)}{d\tau} d\tau.$$

Show that Duhamel's superposition integral can be written as

$$y(t) = \int_0^\infty h(t-\tau)u(\tau) d\tau,$$

which is the convolution integral that we derived in Lecture S5, in the case where  $u(t) = 0$  for  $t < 0$ .

[Hint: Use integration by parts. Show all of your working.]

- (b) An LTI system has impulse response  $h(t) = e^{-\beta t}\sigma(t)$ . Using the convolution integral, find the response  $y(t)$  of the system when the input is  $u(t) = e^{-\alpha t}\sigma(t)$ . Be sure to consider both the case  $\alpha = \beta$  and  $\alpha \neq \beta$ .

## Problem 2: Look-back to Lectures 4 and 5 (15 points)

Consider the simple RC circuit that we analyzed in Lecture S4. In lecture, we determined that the response of the circuit to a unit step input is

$$h_s(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t/RC}, & t > 0. \end{cases}$$

- (a) What is  $h(t)$ , the impulse response of the circuit? Sketch a plot of the impulse response.
- (b) Using the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)u(\tau) d\tau,$$

consider the case  $u(t) = \sigma(t)$ . Compute the response  $y(t)$  via convolution and show that you recover the step response we computed in class.

- (c) Now consider the case where the circuit parameters are  $R = 1$ ,  $C = \frac{1}{2}$  and the input is  $u(t) = e^{-t}\sigma(t)$ . Sketch a plot of  $u(\tau)$  and  $h(t - \tau)$  versus  $\tau$  for the following three cases:
- (i)  $t < 0$
  - (ii)  $t = 0$
  - (iii)  $t > 0$

[Note: The 18.03 Convolution (Flip and Drag) Mathlet might help you here.]

- (d) Using the convolution integral, compute the response  $y(t)$  for the case  $u(t) = e^{-t}\sigma(t)$ ,  $R = 1$ , and  $C = \frac{1}{2}$ .
- (e) Repeat your analysis in part (d), but this time solve the differential equation directly. i.e., solve

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t)$$

in the case  $u(t) = e^{-t}\sigma(t)$ ,  $R = 1$ , and  $C = \frac{1}{2}$ . Compare your answer with that obtained in part (d).

### Problem 3: Look-ahead to Lecture 8 (15 points)

(a) Explain why

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)u(\tau) d\tau = \int_{-\infty}^{\infty} h(\tau)u(t - \tau) d\tau.$$

(b) In the case of sinusoidal inputs, we write  $u(t) = e^{j\omega t}$ , where  $\omega$  is the frequency in rad/s of the forcing. Show that the response of the system in this case is

$$y(t) = e^{j\omega t} \left[ \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau} d\tau \right].$$

(c) We have seen that our first-order LTI systems (e.g. circuit, rocket, roll dynamics) have a step response of the general form

$$h_s(t) = c_1 (1 - e^{-\alpha t}) \sigma(t),$$

where  $c_1$  and  $\alpha$  are constants. This gives us a general impulse response for these systems of the form

$$h(t) = c_2 e^{-\alpha t} \sigma(t),$$

where again  $c_2$  and  $\alpha$  are constants.

What is the response,  $y(t)$ , of such a system to sinusoidal forcing  $u(t) = e^{j\omega t} \sigma(t)$ ?

(d) If  $\alpha > 0$  and  $\omega > 0$ , what is the form of  $y(t)$  for large  $t$ ?

(e) Plot  $y(t)$  for  $0 < t < 6\pi$  in the case  $c_2 = 1$ ,  $\alpha = 0.5$  and  $\omega = 2$ . Plot the real and imaginary parts of  $y(t)$  on different plots. It is suggested that you use Matlab to make these plots (you will also be using Matlab to make plots for the next Systems Lab). Explain the shape of your plot, referring to your analytical results from parts (c) and (d).

Here are some Matlab commands that will be useful:

<code>1+j*2</code>	The complex number $1 + 2j$ .
	Note that you can use either <code>j</code> or <code>i</code> for $\sqrt{-1}$ .
<code>t = [0:0.01:6*pi]</code>	Defines a vector <code>t</code> that goes from 0 to $6\pi$ in steps of 0.01.
<code>v = exp(2*t)</code>	Computes a vector <code>v</code> corresponding to the vector <code>t</code> , where $v = e^{2t}$ .
<code>v = exp(2*j*t)</code>	Computes a vector <code>v</code> corresponding to the vector <code>t</code> , where $v = e^{j2t}$ .
<code>plot(t,y)</code>	Plots <code>y</code> versus <code>t</code> .
<code>grid</code>	Puts a grid on the plot.
<code>xlabel('t')</code>	Sets the x-axis label to be <code>t</code> .
<code>ylabel('y(t)')</code>	Sets the y-axis label to be <code>y(t)</code> .
<code>title('My Plot')</code>	Sets the plot title.
<code>real(y)</code>	Computes the real part of the vector <code>y</code> .
<code>imag(y)</code>	Computes the imaginary part of the vector <code>y</code> .
<code>whos</code>	Tells you what variables you have defined in the Matlab workspace, along with their dimensions.

Note: putting a semicolon at the end of any command suppresses the printout to the screen.